

PACKING INTERVAL GRAPHS WITH VERTEX-DISJOINT TRIANGLES

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Abstract. We show that there exists a polynomial algorithm to pack interval graphs with vertex-disjoint triangles.

1 Introduction

Finding the maximal number of vertex-disjoint triangles in a graph is a well-known NP-complete problem. Whether there exists a polynomial algorithm that solves this problem for interval graphs has been open for a long time.

The packing problem is NP-complete for chordal graphs [2,3]. Interestingly, the question whether a chordal graph can be partitioned into vertex-disjoint triangles can be answered in polynomial time [2].

For splitgraphs the packing problem can be solved via maximum matching. There exists a polynomial algorithm that solves the packing problem for unit interval graphs [5].

In this note we show that the packing problem can be solved in polynomial time for the class of interval graphs.

2 Packing interval graphs

A graph $G = (V, E)$ is an interval graph if and only if it has a consecutive clique arrangement [4]. That is a linear arrangement $\sigma = [C_1, \dots, C_t]$ of the maximal cliques in G such that for each vertex x , the maximal cliques that contain x are consecutive in σ .

Theorem 1. *There exists a polynomial algorithm that computes a triangle packing in interval graphs.*

Proof. Let G be an interval graph and let

$$\sigma = [C_1, \dots, C_t] \tag{1}$$

be a consecutive clique arrangement for G . Write n for the number of vertices in G . Notice that $t \leq n$ where n is the number of vertices in G , since a chordal graph with n vertices has at most n maximal cliques.

Consider a triangle packing \mathcal{T} . A vertex x is covered if $x \in T$ for some $T \in \mathcal{T}$. Notice that in any maximal packing, every maximal clique in G has at most two vertices that are not covered.

We describe an algorithm that computes a triangle packing via dynamic programming. For $i \in \{1, \dots, t\}$, let

$$\sigma_i = [C_1, \dots, C_i] \quad (2)$$

and let G_i be the subgraph of G induced by the maximal cliques in σ_i .

For every $i \in \{1, \dots, t\}$ we keep the following invariant. Let $S \subseteq C_i$ with $|S| \leq 2$. If there is a triangle packing in G_i that covers all vertices of C_i except those in S , then $f_i(S)$ is the maximal number of triangles in such a triangle packing. If there is no such triangle packing in G_i then $f_i(S) = 0$.

First consider $i = 1$. Any maximal triangle packing for G_1 consists of

$$\left\lfloor \frac{|C_1|}{3} \right\rfloor \quad (3)$$

triangles. Let

$$k = |C_1| - 3 \cdot \left\lfloor \frac{|C_1|}{3} \right\rfloor = |C_1| \bmod 3 \quad (4)$$

be the number of vertices that are not covered by triangles.

By definition, we have that

$$f_1(S) = \begin{cases} \left\lfloor \frac{|C_1|}{3} \right\rfloor & \text{if } |S| = k, \text{ and} \\ 0 & \text{otherwise,} \end{cases} \quad (5)$$

where k is given by Equation 4.

Consider the transition from i to $i + 1$. Let $S \subseteq C_{i+1}$ with $|S| \leq 2$. We claim that

$$f_{i+1}(S) = \max_{S'} f_i(S') + \kappa \quad (6)$$

where

(a)

$$S' \subseteq C_i \quad \text{and} \quad |S'| \leq 2 \quad \text{and} \quad S \cap C_i \subseteq S' \cap C_{i+1} \quad \text{and}$$

(b) $S'' = (S' \cap C_{i+1}) \setminus S$, and

(c)

$$\kappa = \frac{|C_{i+1} \setminus (C_i \cup S) \cup S''|}{3} \quad \text{is integer.}$$

Obviously, the right-hand side of Equation (6) is a lowerbound for $f_{i+1}(S)$. We show that it is also an upperbound.

Consider a maximum triangle packing \mathcal{T}_{i+1} for G_{i+1} . Let $S \subseteq C_{i+1}$ be the set of vertices that are not covered by \mathcal{T}_{i+1} . Let $\mathcal{T}_i \subseteq \mathcal{T}_{i+1}$ be the set of triangles that have all vertices in G_i and let S' be the set of vertices in C_i that are not covered by \mathcal{T}_i .

We show that we may assume that $|S'| \leq 2$. Assume $|S'| \geq 3$. Let α , β , and γ be three vertices of S' that are covered by triangles in \mathcal{T}_{i+1} . First assume that $\{\alpha, \beta, p\}$ and $\{\gamma, q, r\}$ are triangles of \mathcal{T}_{i+1} with p, q and r in $C_{i+1} \setminus C_i$. Then replace these triangles with $\{\alpha, \beta, \gamma\}$ and $\{p, q, r\}$. Now assume that $\{\alpha, p, q\}$, $\{\beta, r, s\}$ and $\{\gamma, u, v\}$ are three triangles of \mathcal{T}_{i+1} with p, q, r, s, u and v in $C_{i+1} \setminus C_i$. Then replace the three triangles by $\{\alpha, \beta, \gamma\}$, $\{p, q, r\}$ and $\{s, u, v\}$.

Now assume that there are exactly two vertices α and β in S' that are covered by triangles $\{\alpha, p, q\}$ and $\{\beta, r, s\}$ in \mathcal{T}_{i+1} with p, q, r and s in $C_{i+1} \setminus C_i$. Then we may replace these triangles in \mathcal{T}_{i+1} with $\{\alpha, \beta, p\}$ and $\{q, r, s\}$. Assume that there exists a vertex $z \in S' \setminus \{\alpha, \beta\}$. Then replace $\{\alpha, \beta, p\}$ with $\{\alpha, \beta, z\}$.

Assume that there exists exactly one vertex α in S' which is covered by a triangle $\{\alpha, p, q\} \in \mathcal{T}_{i+1}$ with p and q in $C_{i+1} \setminus C_i$. Assume that there are two vertices y and z in $S' \setminus \{\alpha\}$. Then replace the triangle $\{\alpha, p, q\}$ in \mathcal{T}_{i+1} with $\{\alpha, y, z\}$.

The replacements above don't change the number of triangles in \mathcal{T}_{i+1} . In all cases $|S'|$ decreases, and so this proves the claim.

The vertices of $C_{i+1} \setminus (C_i \cup S) \cup S''$ are in triangles that have at least one vertex in $C_{i+1} \setminus C_i$, where S'' is defined as in item (b). The number of these triangles is κ , as defined in item (c). Thus κ must be an integer.

We show that this algorithm runs in polynomial time. For the computation of f_{i+1} , the algorithm considers $O(|C_{i+1}|^2)$ subsets S . To compute the maximum in Equation (6) the algorithm considers all subsets $S' \subseteq C_i$ with at most two vertices. The table look-up of $f_i(S')$ and the check if the choice of S and S' yields an integer κ as in item (c) take constant time. The final answer is obtained by looking up the maximal value $f_t(S)$ over $S \subseteq C_t$ with $|S| \leq 2$. This shows that the algorithm can be implemented to run in time proportional to

$$\sum_{i=1}^{t-1} |C_i|^2 \cdot |C_{i+1}|^2 + |C_t|^2 = O(n^5).$$

This prove the theorem. □

References

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